

Classical back reaction of low-frequency cosmic gravitational radiation

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We study in a Brill-Hartle type of approximation the back reaction of a superposition of linear gravitational waves on its own mean gravitational field up to second order in the wave amplitudes. The background field is taken as a spatially flat Einstein-de Sitter geometry. In order to follow inflationary scenarios, the wavelengths are allowed to exceed the temporary Hubble distance. As in optical coherence theory, the wave amplitudes are considered as random variables, which form a homogeneous and isotropic stochastic process, sharing the symmetries of the background metric. A segregation of the field equations into equations for the wave amplitudes and equations for the background field is performed by averaging the field equations and interpreting the averaging process as a stochastic (ensemble) average. The spectral densities satisfy a system of ordinary differential equations. The effective stress-energy tensor for the random gravity waves is calculated in terms of correlation functions and covers subhorizon as well as superhorizon modes, where superhorizon modes give in many cases negative contributions to energy density and pressure. We discuss solutions of the second-order equations including pure gravitational radiation universes.

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I. INTRODUCTION

Many classical aspects of gravitational waves are still poorly understood. Most of them are connected with the fact that the full theory of gravitational radiation as following from general relativity is extremely nonlinear. Surprises can be expected, if the nonlinear regime becomes more deeply explored. Pure numerical methods acting on the full field equations [1] are important but cannot be the only key to the nonlinear regime. They should be supplemented by suitable approximation methods, which allow an analytical or semi-analytical approach. Already in 1964 Brill and Hartle [2] proposed a scheme, which takes the back reaction of linear gravitational waves on the background metric into account. Usually, the Brill-Hartle method (see also [3], [4]) is considered as a *high-frequency* approximation for gravitational radiation. For cosmological application the full spectrum of gravitational radiation including low-frequency (superhorizon) modes must be studied, if its generation and propagation through inflationary stages is considered. We discuss in this article a different interpretation of the Brill-Hartle approach, which allows to treat also *low* frequencies.

The Brill-Hartle method requires an average over small scale ripples of the geometry. How averages can be formulated consistently in general relativity, is a basic and still not completely solved problem [5]. Our assumption is to interpret the perturbations of the geometry as *random functions* as in optical coherence theory [6] or in the Monin-Yaglom approach to statistical fluid mechanics [7]. The averages of the Brill-Hartle method are then taken as ensemble averages, not as averages over space-time regions. How stochastic averaging and a random geome-

try can be reconciled with general relativity in a rigorous way is also an open question. The problem is not touched upon here, we adopt instead a field theoretical approach in the sense of the Monin-Yaglom treatment.

Along this line we discuss the response of the expansion rate and of the wave amplitudes to the mean gravitational field produced by the waves. We assume that the stochastic process for the wave amplitudes is homogeneous and isotropic. It is not necessary to suppose it to be Gaussian, since only two-point correlation functions are involved at the approximation level considered here. The background is kept as simple as possible, a flat 3-geometry with scale factor $a(\eta)$ and conformal time η is assumed. The treatment is completely classical, we do not consider back reaction effects from quantum gravity [8], semiclassical gravity [9] or an Einstein-Langevin equation [10]. With methods similar to those discussed here, the subject was treated in [11] and [12].

The paper is organized as follows. In section II the segregation of the field equations into equations for the wave amplitudes and equations for the background field is discussed and, following earlier treatments [13], [14], the effective stress-energy tensor for the random gravity waves is calculated in terms of correlation functions for the wave amplitudes. Depending on the wave spectrum, major contributions to this tensor may come from waves with wavelengths exceeding the horizon distance. Section III is concerned with gauge problems. The rest of the paper treats solutions of the averaged field equations with various assumptions for the spectrum of gravitational waves and for the presence of matter fields. The aim is to discuss effects of back reaction in general, we do not consider the origin of the waves and will also not take observational constraints into account in this article. In

section IV we consider the evolution of the gravitational wave amplitudes for different frequency regimes in cosmological models, using a number of simplifications. Pure gravitational wave models ("geons") are the subject of Section V. Its discussion is complicated by the necessity to include a non-linear back reaction term in the wave equation. In the final section VI possible improvements of the approach are shortly discussed.

II. AVERAGING THE FIELD EQUATIONS.

We take an Einstein-de Sitter model as background metric and add tensor perturbations, writing* ($i, k = 1, 2, 3$)

$$ds^2 = -a(\eta)^2 d\eta^2 + g_{ik} dx^i dx^k \quad (1)$$

with

$$g_{ik} = a^2 \delta_{ik} + h_{ik}, \quad h_{ii} = 0, \quad h_{ik,k} = 0. \quad (2)$$

It is appropriate to write the field equations to this metric in the (3+1) form, assuming a perfect fluid as matter in the background metric:

$$R^{(3)} + K^2 - K_{ik} K^{ik} = 16\pi G \rho_m, \quad (3)$$

$$g^{kl} K_{ki|l} - K_{,i} = 0, \quad (4)$$

$$-K'_{ik}/a + R^{(3)}_{ik} + K K_{ik} - 2K_{il} K_k^l = 4\pi G g_{ik} (\rho_m - p_m). \quad (5)$$

The covariant derivatives are denoted by a prime and taken with respect to g_{ik} , and $' = \frac{d}{d\eta}$. Indices are moved with the three-metric g_{ik} . $R^{(3)}_{ik}$ and $R^{(3)}$ are the three-dimensional Ricci tensor and scalar for g_{ik} . The extrinsic curvature K_{ik} is given by

$$K_{ik} \equiv -\frac{1}{2a} g'_{ik} = -a' \delta_{ik} - \frac{1}{2a} h'_{ik}. \quad (6)$$

Up to second order we have

$$g^{ik} = \delta_{ik}/a^2 - h_{ik}/a^4 + h_{il} h_{kl}/a^6. \quad (7)$$

With the decompositions (2) and (7) one can write down the field equations explicitly, including all terms up to the second order. Simplification results from the gauge restrictions for h_{ik} , which are taken into account. The Ricci tensor $R^{(3)}_{ik}$ is given by

$$\begin{aligned} R^{(3)}_{ik} = & -\frac{1}{2a^2} h_{ik,ll} \\ & + \frac{1}{2a^4} h_{lm} (h_{ik,lm} + h_{lm,ik} - h_{il,km} - h_{kl,im}) \\ & + \frac{1}{4a^4} h_{lm,i} h_{lm,k} + \frac{1}{2a^4} (h_{il,m} h_{kl,m} - h_{il,m} h_{km,l}), \end{aligned} \quad (8)$$

and the field equations read explicitly

$$\begin{aligned} & 6 \frac{a'^2}{a^4} \\ & + \frac{1}{a^6} (h_{kl} h_{kl,mm} + \frac{3}{4} h_{kl,m} h_{kl,m} - \frac{1}{2} h_{kl,m} h_{km,l}) \\ & - \frac{a'}{a^7} h_{kl} h'_{kl} + 3 \frac{a'^2}{a^8} h_{kl} h_{kl} - \frac{1}{4a^6} h'_{kl} h'_{kl} = 16\pi G \rho_m, \end{aligned} \quad (9)$$

$$\begin{aligned} & - \frac{3a'}{2a^6} h_{kl} h_{kl,i} - \frac{a'}{a^6} h_{kl} h_{ik,l} + \frac{1}{2a^5} h_{kl} h'_{ik,l} \\ & - \frac{1}{2a^5} h_{kl} h'_{kl,i} + \frac{1}{4a^5} h'_{kl} h_{kl,i} - \frac{1}{2a^5} h_{kl,i} h'_{kl} = 0, \end{aligned} \quad (10)$$

$$\begin{aligned} & (\frac{a''}{a} + \frac{a'^2}{a^2}) \delta_{ik} + \frac{1}{2a^2} h''_{ik} - \frac{a'}{a^3} h'_{ik} + 2 \frac{a'^2}{a^4} h_{ik} \\ & - 2 \frac{a'^2}{a^6} h_{il} h_{kl} - \frac{1}{2a^4} h'_{il} h'_{kl} + \frac{a'}{a^5} h'_{il} h_{kl} + \frac{a'}{a^5} h'_{kl} h_{il} \\ & - \frac{a'}{2a^5} \delta_{ik} h'_{lm} h_{lm} + \frac{a'^2}{a^6} \delta_{ik} h_{lm} h_{lm} + R^{(3)}_{ik} \\ & = 4\pi G g_{ik} (\rho_m - p_m). \end{aligned} \quad (11)$$

The Brill-Hartle method usually starts with the assumption that the space-time variation of the small ripples $h_{\mu\nu}$ (of order h/λ , where λ is a typical radiation wavelength) is much larger than the variation of the background metric (of order $1/L$). Thus terms in the Ricci tensor which are bilinear in $h_{\mu\nu,\rho}$ (of order $\frac{h^2}{\lambda^2}$) are comparable to terms involving the background metric (of order $\frac{1}{L^2}$), if $h \approx \frac{\lambda}{L}$. This latter condition excludes low-frequency waves $\lambda \geq L$, since h must be sufficiently small to allow a second-order approximation. Terms linear in $h_{\mu\nu,\rho\sigma}$ are much larger and should therefore vanish separately, giving rise to the linear wave equation for $h_{\mu\nu}$.

We do not follow this bookkeeping (it was criticized in [16]). Instead, the functions h_{ik} are interpreted as *random functions*. Performing a stochastic average of (9,10,11) removes terms linear in h_{ik} , but keeps terms bilinear in h_{ik} and in the derivatives of h_{ik} . This allows a segregation of the field equations without restrictions for the wavelengths. The result of the stochastic average can formally be written as (Eqn.(10) reduces to an identity)

$$3 \frac{a'^2}{a^4} = 8\pi G (\rho_m + \rho_g), \quad (12)$$

$$\frac{a''}{a} + \frac{a'^2}{a^2} = 4\pi G a^2 (\rho_m - p_m + \rho_g - p_g), \quad (13)$$

where ρ_g and p_g are the averages over nonlinear terms. It is convenient to interpret ρ_g, p_g as the effective density

*We follow the conventions by Thorne, Misner and Wheeler [15].

and pressure of the gravitational radiation field. Subtracting the averaged field equation (13) (multiplied with δ_{ik}) from (11) and neglecting higher-order terms of the type $\langle A \rangle - A$, as is usually done in a self-consistent field approximation, one obtains a *modified* wave equation for the amplitudes h_{ik}

$$h''_{ik} - \Delta h_{ik} - 2\frac{a'}{a}h'_{ik} + 2h_{ik}\left(\frac{a'^2}{a^2} - \frac{a''}{a} + b\right) = 0 \quad (14)$$

with

$$b = 4\pi\zeta Ga^2(\rho_g - p_g). \quad (15)$$

This equation differs from the usual form by a time-dependent term b representing a back reaction of the energy and pressure of the waves on their propagation (a factor of ζ was introduced in front of b in order to switch off the back reaction term for comparison purposes). Since ρ_g and p_g depend on solutions of the wave equation, some nonlinearity is thus introduced. b results from second-order bilinear terms and is therefore neglected in linear treatments of cosmic wave propagation, but should be kept in the spirit of our approach. In situations where the waves do not appreciably influence the scale factor evolution, the back reaction term can be neglected. Also its influence is usually small in the high-frequency approximation, when the wavelengths are small compared to the Hubble distance. Note also that the back reaction term is exactly zero, if the wave background has the Zeldovich equation of state $p_p = \rho_g$. On the other hand, if low-frequency radiation contributes appreciably to the average density and pressure and hence to the scale factor evolution, the modified wave equation must be considered in general. (A previous paper ([17]) on the same subject was based on the linear wave equation with $b = 0$. We shall continue its use in Sec. IV in order to compare later the results with those of the general case $b \neq 0$).

Some authors use different definitions of the wave amplitudes by applying factors of a on them. We have defined the spatial components h_{ik} as perturbations to the three-metric $a^2\delta_{ik}$. Equivalent to (14) are

$$(h_{ik}/a^2)'' - \Delta(h_{ik}/a^2) + 2\frac{a'}{a}(h_{ik}/a^2)' + 2bh_{ik}/a^2 = 0 \quad (16)$$

and

$$(h_{ik}/a)'' - \Delta(h_{ik}/a) + (2b - \frac{a''}{a})(h_{ik}/a) = 0, \quad (17)$$

which are sometimes easier to use. It was frequently noted that gravitational wave perturbations in a Friedman-Robertson-Walker (FRW) universe may be described as a pair of massless minimally coupled scalar fields in the same background space-time (see, e.g., Ford

and Parker in ([9])). If back reaction is important, this correspondence is lost in general, but for a de Sitter scale factor $a \sim 1/\eta$ the modified wave equation has the form $(\square + \xi R + m^2)(\Phi/a) = 0$ with $\xi = \frac{1}{2} - \frac{1}{6}m^2a^2\eta^2$, which characterizes non-minimally coupled and possibly massive scalar particles.

To calculate the components of the gravitational stress-energy tensor, we assume that the random process represented by h_{ik} is homogeneous and isotropic. The correlation functions which enter (12,13) are related to certain spectral densities. To see this, we represent h_{ik} as stochastic Fourier integral (see [7] for a detailed treatment of the spectral representation of random processes):

$$h_{ik}(\mathbf{x}, \eta) = \int \gamma_{ik}(\mathbf{k}, \eta) e^{i\mathbf{k}\mathbf{x}} d\mathbf{k} + \text{conj.compl.} \quad (18)$$

From (14), the Fourier amplitudes satisfy the ordinary differential equation

$$\gamma''_{ik} - 2\frac{a'}{a}\gamma'_{ik} + \gamma_{ik}(2\frac{a'^2}{a^2} - 2\frac{a''}{a} + k^2 + 2b) = 0. \quad (19)$$

Amplitudes of the correlation functions may now be written as frequency integrals over spectral densities:

$$\langle h_{ik}(\mathbf{x}, \eta) h_{lm}(\mathbf{x}, \eta) \rangle = \int \langle (\gamma_{iklm} + \gamma_{iklm}^*) \rangle d\mathbf{k}, \quad (20)$$

where[†] [7]

$$\langle \gamma_{ik}(\mathbf{k}, \eta) \gamma_{lm}^*(\tilde{\mathbf{k}}, \eta) \rangle = \delta(\mathbf{k} - \tilde{\mathbf{k}}) \gamma_{iklm}, \quad (21)$$

$$\langle \gamma_{ik}(\mathbf{k}, \eta) \gamma_{lm}(\tilde{\mathbf{k}}, \eta) \rangle = 0. \quad (22)$$

It is easy to see that the spectral densities γ_{iklm} satisfy the symmetry relations

$$\gamma_{iklm}^* = \gamma_{lmik}, \quad \gamma_{iklm} = \gamma_{kil m}, \quad (23)$$

$$\gamma_{ikll} = 0, \quad \gamma_{iklm} k^m = 0. \quad (24)$$

The general solution of the algebraic constraints (23, 24) for the spectral densities contains four complex functions of k and η , which describe polarized background radiation in general. The stochastic background of gravitational waves expected from pre-galactic stages of the Universe could be polarized due to strong anisotropies expected at Planck time. Here we confine the discussion to unpolarized radiation, represented by a single (real) spectral function $\alpha(k, \eta)$. Then γ_{iklm} can be written as:

$$\gamma_{iklm} = \alpha(k, \eta) \bar{\delta}_{iklm}, \quad (25)$$

$$\bar{\delta}_{iklm} = \bar{\delta}_{il} \bar{\delta}_{km} + \bar{\delta}_{im} \bar{\delta}_{kl} - \bar{\delta}_{ik} \bar{\delta}_{lm}, \quad (26)$$

[†]We refer to equal space - equal time correlators of the form (20) as "correlation functions".

where

$$\bar{\delta}_{ik} = \delta_{ik} - k_i k_k / k^2, \quad k^2 = k^l k^l. \quad (27)$$

The transversal δ -symbol satisfies

$$\bar{\delta}_{ii} = 2, \quad \bar{\delta}_{ik} k^k = 0, \quad \bar{\delta}_{il} \bar{\delta}_{kl} = \bar{\delta}_{ik}. \quad (28)$$

To calculate the averaged stress-energy tensor, we need also the spectral densities

$$\langle \gamma'_{ik} \gamma'^*_{lm} \rangle = \beta \delta(\mathbf{k} - \tilde{\mathbf{k}}) \bar{\delta}_{iklm}, \quad (29)$$

$$\langle \gamma_{ik} \gamma'^*_{lm} \rangle = \gamma \delta(\mathbf{k} - \tilde{\mathbf{k}}) \bar{\delta}_{iklm}, \quad (30)$$

where β is real and γ is complex in general (the correlation functions α , β and γ were introduced in [14]). Similar to (20), the correlation functions which are needed to calculate the effective wave stress-energy tensor can be written as

$$\langle h_{ik,r} h_{lm,s} \rangle = 2 \int \alpha \bar{\delta}_{iklm} k_r k_s d\mathbf{k},$$

$$\langle h_{ik} h_{lm,rs} \rangle = -\langle h_{ik,r} h_{lm,s} \rangle,$$

$$\langle h'_{ik} h'_{lm} \rangle = 2 \int \beta \bar{\delta}_{iklm} d\mathbf{k},$$

$$\langle h_{ik} h'_{lm} \rangle = 2 \int \Re(\gamma) \bar{\delta}_{iklm} d\mathbf{k},$$

$$\langle h_{ik} h_{lm,r} \rangle = 2 \int \alpha k_r \bar{\delta}_{iklm} d\mathbf{k},$$

$$\langle h_{ik} h'_{lm,r} \rangle = 2 \int \Im(\gamma) \bar{\delta}_{iklm} k_r d\mathbf{k}$$

$$\langle h'_{ik} h_{lm,r} \rangle = -\langle h_{ik} h'_{lm,r} \rangle.$$

Since α, β, γ are functions of k (and η) only, the angular integrations can be performed easily. This gives with $p_{iklm} \equiv \delta_{il} \delta_{km} + \delta_{im} \delta_{kl}$:

$$\begin{aligned} \langle h_{ik} h_{lm} \rangle &= \frac{16\pi}{15} (3p_{iklm} - 2\delta_{ik} \delta_{lm}) \int \alpha k^2 dk, \\ \langle h_{ik,r} h_{lm,s} \rangle &= \frac{16\pi}{105} (-10\delta_{ik} \delta_{lm} \delta_{rs} + 11p_{iklm} \delta_{rs} \\ &\quad - 3p_{ilrs} \delta_{km} - 3p_{lkrs} \delta_{im} \\ &\quad - 3p_{kmrs} \delta_{il} - 3p_{imrs} \delta_{kl} \\ &\quad + 4p_{ikrs} \delta_{lm} + 4p_{lmrs} \delta_{ik}) \int \alpha k^4 dk, \\ \langle h_{ik} h_{lm,rs} \rangle &= -\langle h_{ik,r} h_{lm,s} \rangle \\ \langle h'_{ik} h'_{lm} \rangle &= \frac{16\pi}{15} (3p_{iklm} - 2\delta_{ik} \delta_{lm}) \int \beta k^2 dk, \\ \langle h_{ik} h'_{lm} \rangle &= \frac{16\pi}{15} (3p_{iklm} - 2\delta_{ik} \delta_{lm}) \int \Re(\gamma) k^2 dk, \\ \langle h_{ik} h_{lm,r} \rangle &= 0, \\ \langle h'_{ik} h_{lm,r} \rangle &= 0, \\ \langle h_{ik} h'_{lm,r} \rangle &= 0. \end{aligned}$$

Since only the real part of γ enters the averaged expressions, we denote subsequently this real part by γ .

Some unusual results can be expected when one deals with stochastic averages over nonlinear quantities. For the averaged three-dimensional Ricci tensor we obtain, using the just derived relations,

$$\langle R^{(3)}_{ik} \rangle = \frac{8\pi}{3a^4} \delta_{ik} \int \alpha k^4 dk. \quad (31)$$

Thus, whereas the three-dimensional background metric is flat, a random superposition of gravitational radiation to this background produces an averaged 3-Ricci tensor with positive curvature, if curvatures are defined as eigenvalues of the Ricci tensor. On the other hand, the stochastic average of the 3-curvature scalar $R^{(3)} = g^{ik} R^{(3)}_{ik}$ is negative:

$$\langle R^{(3)} \rangle = -\frac{8\pi}{a^6} \int \alpha k^4 dk. \quad (32)$$

The bilinear terms which enter $R^{(3)}$ and produce a nonzero stochastic average are different from those entering $R^{(3)}_{ik}$, thus the result $\langle R_{ik} \rangle \langle g^{ik} \rangle \neq \langle R_{ik} g^{ik} \rangle$ does not come as a surprise. But it makes it hard to interpret stochastic averages geometrically.

Energy density and pressure of the gravitational waves are given by similar integrals

$$\rho_g = \frac{1}{2Ga^6} \int dk k^2 (k^2 \alpha + \beta + 4\gamma \frac{a'}{a} - 12\alpha \frac{a'^2}{a^2}), \quad (33)$$

$$p_g = \frac{1}{6Ga^6} \int dk k^2 (7k^2 \alpha - 5\beta + 20\gamma \frac{a'}{a} - 20\alpha \frac{a'^2}{a^2}). \quad (34)$$

The deviation from the HF equation of state follows as

$$\rho_g - 3p_g = \frac{1}{Ga^6} \int dk k^2 (-3k^2 \alpha + 3\beta - 8\gamma \frac{a'}{a} + 4\alpha \frac{a'^2}{a^2}), \quad (35)$$

and the back reaction function b is given by

$$b = \frac{8\pi\zeta}{3a^4} \int dk k^2 (-k^2 \alpha + 2\beta - 2\gamma \frac{a'}{a} - 4\alpha \frac{a'^2}{a^2}). \quad (36)$$

The time evolution of the spectral functions α , β and γ is obtained from (19). Differentiating (21), (29), (30) with respect to η and using (19) leads to a system of coupled differential equations:

$$\alpha' = 2\gamma, \quad (37)$$

$$\beta' = 4\frac{a'}{a}\beta + 2\gamma(2\frac{a''}{a} - 2\frac{a'^2}{a^2} - 2b - k^2), \quad (38)$$

$$\gamma' = 2\frac{a'}{a}\gamma + \beta + \alpha(2\frac{a''}{a} - 2\frac{a'^2}{a^2} - 2b - k^2). \quad (39)$$

With given initial values the time evolution of all spectral densities and hence correlation functions follows from this system, if the scale factor is known. The system for α, β, γ is actually nonlinear due to the presence of the b

term. It can be transformed into a single nonlinear differential equation for α alone. Substituting $\alpha'/2$ for γ in the last two equations and introducing a new function ϵ instead of β by means of

$$\epsilon = \beta - \frac{\alpha'^2}{4\alpha}, \quad (40)$$

one obtains

$$\epsilon' + \left(\frac{\alpha'}{\alpha} - 4\frac{a'}{a}\right)\epsilon = 0. \quad (41)$$

This is integrated to

$$\epsilon = \frac{\epsilon_0(k)a^4}{\alpha}. \quad (42)$$

Then the basic spectral function α satisfies the nonlinear differential equation

$$2\alpha\alpha'' - \alpha'^2 - 4\epsilon_0a^4 - 4\frac{a'}{a}\alpha\alpha' - 4\alpha^2\left(2\frac{a''}{a} - 2\frac{a'^2}{a^2} - 2b - k^2\right) = 0. \quad (43)$$

If we define wave amplitudes with a different power of a , for instance as in (17), this equation can be simplified. With $\alpha = fa^2$,

$$2ff'' - f'^2 + 4f^2(k^2 + 2b - \frac{a''}{a}) - 4\epsilon_0 = 0. \quad (44)$$

This equation is also nonlinear in f , but its nonlinearity arises because we deal with the spectral density of an expression which is *quadratic* in the random wave amplitudes h_{ik} . The solutions of (44) are related to those of a differential equation, which is linear in the wave amplitudes. Let h_1, h_2 be a pair of real solutions to

$$h'' + h(k^2 + 2b - a''/a) = 0 \quad (45)$$

such that

$$h_1h_2' - h_2h_1' = \sqrt{\epsilon_0}, \quad (46)$$

where ϵ_0 is a time-independent function of k only. Then $f = h_1^2 + h_2^2$ is a solution of (44). Not surprisingly, the linear differential equation for h is the Fourier transform of the wave equation (17) for a single realization of the random process. We also note that the solutions of (44) can be reduced to solutions $f_{hom}(k, \eta)$ of the homogeneous part of (44) ($\epsilon_0 = 0$): Any solution of the full equation (44) can be written in terms of a suitable function f_{hom} as

$$f = f_{hom} + \epsilon_0(k)f_{hom} \left(\int \frac{d\eta}{f_{home}} \right)^2. \quad (47)$$

It is convenient to write the expressions for ρ and p and for other stochastic averages in terms of four frequency-independent, but in general time-dependent, integrals

over the spectral density $f(k, \eta)$, denoted as "moments" subsequently:

$$f_0 = \int dk k^2 \frac{\epsilon_0(k)}{f(k, \eta)}, \quad f_1 = \int dk k^2 \frac{f'^2(k, \eta)}{f(k, \eta)}, \\ f_2 = \int dk k^2 f(k, \eta), \quad f_4 = \int dk k^4 f(k, \eta). \quad (48)$$

In the density and pressure equation, f_0 and f_1 appear only in the combination

$$g_4 = f_0 + f_1/4. \quad (49)$$

We also note the useful relations

$$g_4' = -f_4' + \left(\frac{a''}{a} - 2b\right)f_2', \quad (50)$$

$$f_2'' = 2g_4 + 2\left(\frac{a''}{a} - 2b\right)f_2 - 2f_4, \quad (51)$$

which follow from differentiating f_1 once and f_2 twice and using the differential equation for f .

For a general scale factor, energy density and pressure may be rewritten as

$$\rho_g = \frac{1}{2Ga^4}(f_4 + g_4 + 3\frac{a'}{a}f_2' - 7\frac{a'^2}{a^2}f_2), \quad (52)$$

$$3p_g = \frac{1}{2Ga^4}(7f_4 - 5g_4 + 5\frac{a'}{a}f_2' - 5\frac{a'^2}{a^2}f_2). \quad (53)$$

The back reaction function b is given by

$$b = \frac{8\pi\zeta}{3a^2}(-f_4 + 2g_4 + \frac{a'}{a}f_2' - 4\frac{a'^2}{a^2}f_2), \quad (54)$$

thus the relations (50),(51) for the moments become nonlinear in general.

For completeness, we give the ensemble averages of some other geometrical quantities. The mean value of the four-dimensional Ricci scalar can be written as

$$\langle R \rangle = \frac{6a''}{a^3} + \frac{8\pi}{a^4}(3f_4 - 3g_4 + \frac{a'^2}{a^2}f_2 + \frac{a'}{a}f_2'), \quad (55)$$

where the first term is the background contribution. The second term may also be written as $8\pi G(3p_g - \rho_g)$. Since $R = 0$ in vacuum, also the stochastic average of R must vanish. Then (55) is equivalent to a combination of (12) and (13) (with $\rho_m = p_m = 0$), as one can check easily. There are only two independent and in general nonzero components of the averaged Riemann tensor:

$$\langle R_{0101} \rangle = a'^2 - aa'' + \frac{8\pi}{3}(g_4 + \frac{a'^2}{a^2}f_2 - \frac{a'}{a}f_2'), \quad (56)$$

$$\langle R_{1212} \rangle = a'^2 + \frac{4\pi}{3}(3f_4 - g_4 - \frac{a'^2}{a^2}f_2 - \frac{a'}{a}f_2'). \quad (57)$$

The one independent component of the averaged Weyl tensor

$$\langle C_{0101} \rangle = \langle C_{1212} \rangle = \frac{16\pi}{3} (f_4 - \frac{a'^2}{a^2} f_2 + \frac{a'}{2a} f_2'). \quad (58)$$

contains, of course, no background term. All curvature components result from averaging nonlinear terms, they have no direct connection to the first order perturbations of the curvature tensor, whose average is zero. It is immediately seen that $\langle g^{\alpha\beta} \rangle \langle C_{\alpha\mu\beta\nu} \rangle$ differs from zero for some components, while, of course, the original expression $g^{\alpha\beta} C_{\alpha\mu\beta\nu}$ and hence its average vanishes. Averaging the four local curvature invariants connected with the Weyl tensor is more complicated: Apart from the correlation functions listed above further expressions such as $\langle h_{ij,kl} h_{mn,rs} \rangle$ must be calculated, which introduces higher-order moments such as $f_6 = \int dk k^6 f(k, \eta)$.

We summarize the main results of this section. In order to describe the back reaction of gravitational radiation propagating in a flat Friedman universe upon the scale factor, we have replaced the Einstein field equations by a system of equations for averaged quantities. This system consists (i) of the Friedman equations (12),(13) for a cosmic fluid with density ρ_m and pressure p_m , together with the corresponding quantities ρ_g and p_g for gravitational radiation given by (52),(53), and (ii) of the differential equation (44) for the spectral density $f(k, \eta)$. The latter equation may be replaced partially by differential relations for the moments of f . The aim is to solve these equations for the scale factor and for f simultaneously. One has to keep in mind that the results are only reliable if terms higher than second order in the original field equations can safely be neglected.

III. GAUGE TRANSFORMATIONS

We have chosen to work in a fixed gauge. As recently noted by Unruh [18], there is no basic difference between a gauge invariant approach, where the equations are written in terms of "gauge-independent" variables (see, e.g., [12]), and an approach where the gauges are fixed in some way. It is only important to fix the gauges completely, leaving no room for residual gauge freedom, which could be mistaken as physical degree of freedom. We therefore ask for the coordinate transformations $\bar{x}^\mu = x^\mu + \xi^\mu$ preserving the eight constraints $h_{00} = 0$, $h_{0i} = 0$, $h_{ii} = 0$, $h_{ik,k} = 0$. To linear order in ξ^μ the perturbations transform as

$$\bar{h}_{\mu\nu}(x^\mu) = h_{\mu\nu}(x^\mu) - g_{\mu\nu,\rho}^{(0)} \xi^\rho - g_{\mu\rho}^{(0)} \xi_{,\nu}^\rho - g_{\nu\rho}^{(0)} \xi_{,\mu}^\rho. \quad (59)$$

A simple calculation shows that for a general scale factor the transformations we are looking for must satisfy

$$\xi^0 = 0, \quad \xi^i = \xi^i(x^k), \quad \xi_{,i}^i = 0, \quad \xi_{,kk}^i = 0. \quad (60)$$

The functions ξ^μ therefore do not necessarily form a Killing field, and the first-order perturbations are gauge-dependent in general. The spatial components transform as

$$\bar{h}_{ik}(x^i) = h_{ik}(x^i) - a^2(\xi_{,k}^i + \xi_{,i}^k). \quad (61)$$

It is easy to see that the first-order wave equation (14) is invariant with respect to (61). The transformation of the correlation functions is derived assuming that the vector ξ^i is also a random process with zero mean and independent of the wave process. We outline the procedure. The random process ξ^i must be homogeneous and isotropic as h_{ik} , so its correlation function can be written

$$\langle \xi^i \xi^k \rangle = \int dk (F_1(k) \frac{k^i k^k}{k^2} + F_2(k) \delta_{ik}). \quad (62)$$

$\xi_{,i}^i = 0$ is translated into $F_1(k) = -F_2(k)$ in the spectral region. It is more difficult to write the condition $\xi_{,kk}^i = 0$ directly as condition for the spectral density F_1 , so we first proceed without this constraint. With (61) one forms $\langle \bar{h}_{ik} \bar{h}_{lm} \rangle$, this gives

$$\bar{f}_2 = f_2 + \frac{a^2}{2} \int dk k^4 F_2(k). \quad (63)$$

Similarly, the reduction of $\langle \bar{h}'_{ik} \bar{h}'_{lm} \rangle$ yields

$$\bar{g}_4 = g_4 + \frac{a'^2}{2} \int dk k^4 F_2(k). \quad (64)$$

Writing out $\langle \bar{h}_{ik,r} \bar{h}_{lm,s} \rangle$ shows that the average of the ξ^i -depending terms does not have the symmetries of the other terms. This reflects the fact that the condition $\xi_{,kk}^i = 0$ was not taken into account. We conclude that the spectral density satisfies the integral relation $\int dk k^6 F_2(k) = 0$, and that f_4 transforms as

$$\bar{f}_4 = f_4. \quad (65)$$

It is easily seen that the density ρ_g and pressure p_g as defined by (33),(34) are invariant against the transformations (63)-(65), and this holds also for the averaged components of the Weyl tensor and for the Ricci scalar.

As is well known, the effective stress-energy tensor is *not* gauge invariant with respect to *general* coordinate transformations, i.e., transformations which violate the constraints. However, as shown by Abramo, Brandenberger and Mukhanov ([11], see also [19], [4]), the gauges change at the same time the background geometry to second order, and these changes just compensate the change of the energy-stress tensor.

IV. SOLUTIONS INCLUDING MATTER FIELDS

We now try to find solutions of the equations derived in section II, assuming that apart from gravitational radiation other forms of matter are present, which dominate dynamically during most stages of the cosmic evolution, if not always. Dominance of matter means that the b -term in the wave equation can be neglected.

A. The high-frequency regime

The high-frequency (HF) regime is defined by the assumption that all wavelengths are small compared with the temporary Hubble distance. Then the time derivatives which occur in (37-39) are small: Let A be one of the quantities α, β, γ , then A' is of the order A/T , where T is a Hubble time, $T \approx a/a'$, and this is much smaller than a multiplication with the wave frequency k : $A/T \ll Ak$. The terms β and $k^2\alpha$ in (39) are therefore much larger than other terms and must cancel in a HF approximation:

$$\beta = k^2\alpha. \quad (66)$$

Using this relation and neglecting time derivatives in (33, 34), one obtains

$$\rho_g = 3p_g = \frac{1}{Ga^6} \int dk k^4 \alpha, \quad (67)$$

equivalent to the equation of state for a gas of noninteracting massless particles. A similar cancellation of terms must occur in (43). This requires $\epsilon_0 > 0$, gives

$$\alpha = \sqrt{\epsilon_0} a^2 / k \quad (68)$$

and identifies the function $\epsilon_0(k)$ as closely related to the time-independent spectral function in the HF regime. The energy density in this regime,

$$\rho_g = \frac{1}{Ga^4} \int dk k^3 \sqrt{\epsilon_0}, \quad (69)$$

shows the typical $1/a^4$ dependence on the scale factor for background radiation. Apart from an overall redshift factor, the gravitational wave spectrum does not change in time.

The expression (67) for the effective gravitational energy density and pressure of *high-frequency* gravitational radiation holds also in the case that this radiation contributes appreciably to the background geometry, beside other forms of matter. The result is only a changed time dependence of the scale factor a .

B. Stress tensor evolution in a fixed background

Considering now the full spectrum including both low-frequency (LF) and HF modes, one must return to the solution of the complete equation (44). The change of the spectrum during expansion depends on the wavelength. The solutions f of (44) for power law scale factors can be represented by Hankel functions of second kind, if we neglect the b -term assuming that gravitational waves do not contribute appreciably to the scale factor. We prefer explicit expressions, since this allows to use Fourier methods. For the matter universe ($a \sim \eta^2$), the radiation

universe ($a \sim \eta$) and the de Sitter universe ($a \sim 1/\eta$) as background geometry the general solution can then be written

$$\alpha/a^2 = f = 2npp^* + (l + im)p^2 + (l - im)p^{*2}. \quad (70)$$

l, m, n are three functions of k and connected with ϵ_0 by

$$\epsilon_0 = 4k^2(n^2 - l^2 - m^2). \quad (71)$$

$p(x)$ is a complex function of $x = k\eta$, given by

$$p(x) = (1 + is/x)e^{ix} \quad (72)$$

with $s = 1$ in the matter and de Sitter universe and $s = 0$ in the radiation universe. (In spite of the different scale factors, the spectral density f in the de Sitter cosmos has the same form as in the matter universe, since a''/a is the same. This coincidence holds only for the spectral density f and is destroyed, if one returns to α). For large x , f reaches the same asymptotic form in all models. This asymptotic form coincides with the HF expression $f = \sqrt{\epsilon_0}/k$ considered previously, if the ratios $\frac{l}{n}$ and $\frac{m}{n}$ tend to zero for large k .

How do the different wavelengths contribute to the effective stress-energy tensor? It is usually assumed that modes with wavelengths larger than the temporary Hubble radius can be discarded, since they are unobservable and not true waves [20], have no appreciable influence on the local wave energy density [21], [22], [23], or appear locally as gauge transformation [24], [18]. The relations (52) and (53) allow a definite answer. For a radiation universe $a \sim \eta$ one obtains:

$$a^4 G \rho_g = 2n_4 - \frac{7}{\eta^2} n_2 - \frac{7}{2\eta^2} \psi_2(\eta) + \frac{3}{2\eta} \psi_2'(\eta), \quad (73)$$

$$3a^4 G p_g = 2n_4 - \frac{5}{\eta^2} n_2 - \frac{5}{2\eta^2} \psi_2(\eta) + \frac{5}{2\eta} \psi_2'(\eta) - \frac{3}{2} \psi_2''(\eta) \quad (74)$$

with the time function

$$\psi_2(\eta) = 2 \int_0^\infty k^2 (l(k) \cos(2k\eta) - m(k) \sin(2k\eta)) dk, \quad (75)$$

we also use the notation $n_l = \int_0^\infty n(k) k^l dk$ for the time-independent moments of $n(k)$. The spectral functions l, m, n are constrained by the condition that the frequency integrals (including the moments n_2 and n_4) converge. To justify the assumption of a pre-determined scale factor, the amplitudes must be small enough that their contribution to the background geometry is negligible. If the function $\psi_2(\eta)$ with its first time derivative is bounded for large η and ψ_2'' tends to zero for large η , the relation $p_g = \rho_g/3$ between pressure and density follows asymptotically for large η , as seen from (73),(74). For small times however, ρ_g and p_g show strong deviations

from this relation. In particular, ρ_g as well as p_g become *negative* for sufficiently small η . In the case $l = m = 0$ (and therefore $\psi_2 = 0$) this is immediately seen from (73) and (74). ρ_g and p_g are also negative for small η in more general spectra.

The main reason is that the spectrum for wavelengths exceeding the local horizon scale at η (corresponding to frequencies $k < 1/\eta$) gives a negative contribution to the total values of energy density and pressure. If one goes back in time, the superhorizon modes ultimately dominate the spectrum (we always assume that the spectral density in the ultraviolet region drops sufficiently fast to ensure convergence for $k \rightarrow \infty$). Ultraviolet divergence is a problem of quantum theory and does not concern us here).

Since the HF part of the spectrum satisfies a $\rho_g a^4 = \text{const}$ -law and is therefore not very interesting, we concentrate on the LF tail, which is essential for back reaction effects.

C. Back-reaction solutions in the regular low-frequency limit

In the long-wavelength or LF limit $k \rightarrow 0$ we assume that the spectral density f can be expanded in powers of k around $k = 0$. This assumption excludes a singularity at $k = 0$, which in some cases can be interpreted as leading to a finite infrared contribution to ρ and p (Section V). Writing down (44) explicitly requires to calculate the back reaction term b and hence the integrals in (48). In the sense of our approximation, this is trivial if the integration is assumed to extend to a maximal (but still small) $k = k_1$. Expanding also ϵ_0 , (44) leads to

$$2qq'' - q'^2 + 4qq' \frac{a'}{a} + \frac{32\pi k_1^2 \zeta}{3} q \left(-\frac{1}{5} k_1^2 q^2 + \frac{1}{6} q'^2 + qq' \frac{a'}{a} \right) = 4 \frac{\epsilon_0}{a^4} \quad (76)$$

as lowest-order equation of a hierarchy of equations for the expansion coefficients (we have written q for $\lim_{k \rightarrow 0} f/a^2$ and again ϵ_0 for $\lim_{k \rightarrow 0} \epsilon_0$). Only this lowest-order approximation will be discussed. Since the solution of (44) can be written in the form (47), it is appropriate to treat the case $\epsilon_0 = 0$. Here $q = \text{const}$ is an obvious solution of (76), provided we neglect the term proportional to k_1^2 in the bracket, which is small compared with the other terms in a LF approximation. The effective energy density and pressure of the extreme LF background follows as

$$\rho_g = \frac{3}{7} p_g = \frac{q k_1^5}{10 G a^2}. \quad (77)$$

Note the a^{-2} decay compared to the a^{-4} decay of the energy density of the high-frequency radiation field (cf.

[12]). Thus the influence of the LF gravitational radiation field on the scale factor is small for sufficiently early times compared to other forms of radiation. The gravitational wave background considered here is alone not able to support a cosmological model: Since the time or scale factor dependence of energy density and pressure is already fixed, the two equations (12) and (13) are not compatible. But we may assume that a relativistic fluid with a $p_m = \rho_m/3$ equation of state is also present. This ensures compatibility at the price of fixing the density ratio between fluid and waves and gives

$$\rho_m = \frac{15a'^2 - 4a^2 k_1^5 q \pi}{40 G \pi a^4}, \quad (78)$$

$$a'' + \frac{4}{5} \pi k_1^5 q a = 0. \quad (79)$$

From the second relation the scale factor follows as

$$a = a_1 \sin(l\eta), \quad l^2 = 4\pi k_1^5 q/5, \quad (80)$$

thus the energy density of the relativistic fluid changes as

$$\rho_m = \frac{k_1^5 q (3 - 4 \sin^2(l\eta))}{10 G a_1^2 \sin^4(l\eta)}. \quad (81)$$

It appears that the mere existence of a LF gravitational wave background could cause a reversal of the expansion, however small its energy density may be initially (that is, for times η with $l\eta \ll 1$) compared to that of the relativistic fluid, note that $\rho_g/\rho_m = \sin^2(l\eta)/(3 - 4 \sin^2(l\eta))$. For $l\eta > 0.8861$, ρ_g exceeds ρ_m , for $\eta l > \pi/3$ the fluid energy density becomes formally negative through the interaction with the gravitational wave fields. The averaged Weyl tensor given by

$$\langle C_{0101} \rangle = \langle C_{1212} \rangle = \frac{16\pi a_1^2 k_1^5 q}{15} \sin^2(l\eta) \quad (82)$$

vanishes at the singularity $\eta = 0$ and is non-zero for $\eta > 0$, suggesting that the presence of a LF component of gravitational radiation is not a coordinate effect.

The case of a nonzero limit ϵ_0 for $k \rightarrow 0$ is more complicated and seems not to allow an analytic solution. As another example we treat the interaction of the LF background with a Λ -term. It is common practice [25] to write the Λ -constant on the right-hand-side of the field equations, i.e. to consider a fluid with the components $\rho_m = \frac{\Lambda}{8\pi G}$, $p_m = -\frac{\Lambda}{8\pi G}$. Considering again the case $\lim_{k \rightarrow 0} \epsilon_0 = 0$, we arrive at (77) as before, but the field equations (12) and (13) lead now to an exponential increase or decrease of the scale factor (in the η -coordinate):

$$a = a_1 \exp(\pm \eta \sqrt{4\pi k_1^5 q/3}). \quad (83)$$

As in the former case of a relativistic fluid, the " Λ -matter" and the LF gravitational wave background are coupled since

$$q^3 = \frac{45G\Lambda}{64\pi^2 k_1^3} \quad (84)$$

is required for consistency. The Weyl tensor components

$$\langle C_{0101} \rangle = \langle C_{1212} \rangle = \frac{16\pi a^2 k_1^5 q}{15} \quad (85)$$

are also finite (and nonzero) at $\eta = 0$.

V. DOMINANT GRAVITATIONAL WAVE BACKGROUND

Let us now assume a pure gravitational radiation universe. In the high frequency approximation all is already done: We do not have to face an appreciable back reaction term in the wave equation, the spectral shape of the waves is time-independent, the amplitude decreases only due to redshift effects, the equation of state can well be approximated by $\rho_g = 3p_g$, and (12),(13) (with $\rho_m = p_m = 0$) give the Tolman radiation cosmos.

The presence of only LF components in the spectrum does not allow a solution based on gravitational waves alone, thus the spectrum will not be restricted subsequently. We use the two equations (50) and (51) for the frequency integrated quantities g_4 and f_4 . Adding (12),(13) with $\rho_m = 0$ and $p_m = 0$, one has two further equations to determine the four unknown time functions f_2, f_4, g_4 and a . We solve the last equations for g_4 and f_4 and use the result in (51) to find f_2'' . This leads to

$$g_4 = \frac{1}{8\pi}(aa'' + 3a'^2) + \frac{11}{3}\frac{a'^2}{a^2}f_2 - \frac{4}{3}\frac{a'}{a}f_2', \quad (86)$$

$$f_4 = \frac{1}{8\pi}(-aa'' + 3a'^2) + \frac{10}{3}\frac{a'^2}{a^2}f_2 - \frac{5}{3}\frac{a'}{a}f_2', \quad (87)$$

$$f_2'' = \frac{1}{2\pi}aa'' + (2(1-2\zeta)\frac{a''}{a} + \frac{2}{3}(1-6\zeta)\frac{a'^2}{a^2})f_2 + \frac{2a'}{3a}f_2', \quad (88)$$

$$0 = 2\frac{a'}{a}((3\zeta+2)\frac{a''}{a} + (3\zeta-4)\frac{a'^2}{a^2})f_2 + ((\zeta-2)\frac{a''}{a} + (\zeta+4)\frac{a'^2}{a^2})f_2' \quad (89)$$

This system may be studied in two cases, (i) assuming $\zeta = 0$, that is, no back reaction in the wave equation and (ii) full back reaction, $\zeta = 1$. Furthermore one has to solve the generalized wave equation (14) and check the compatibility of its solution with the moments derived from (86)-(89). Finally, if all this succeeds, the energy density and pressure of the waves follow from the familiar equations

$$\rho_g = 3a'^2/(8\pi Ga^4), \quad p_g = (-2aa'' + a'^2)/(8\pi Ga^4). \quad (90)$$

A. $\zeta = 0$: Tolman universe

With $\zeta = 0$, (89) can be written as the product of two factors:

$$(2a'f_2 - af_2')(aa'' - 2a'^2) = 0. \quad (91)$$

Thus two different cases emerge, depending on which factor vanishes. If the first factor in (91) is zero, one obtains with an integration constant c (which may be gauged to zero)

$$f_2 = ca^2. \quad (92)$$

This relation is compatible with (88) if and only if $a'' = 0$, that is, if the scale factor has the time dependence $a = b\eta$ of the Tolman radiation cosmos [26], with the energy density and pressure given by

$$\rho_g = 3p_g = \frac{3}{8\pi Gb^2\eta^4}. \quad (93)$$

From (86, 87) it follows that the other moments are time-independent:

$$g_4 = b^2(c + \frac{3}{8\pi}), \quad f_4 = \frac{3b^2}{8\pi}. \quad (94)$$

The average of the Weyl tensor component C_{0101} or C_{1212} comes out as the constant $2b^2$. The expressions for f_2, f_4 and g_4 found as solutions of differential equations must be compatible with those derived directly from the spectral density (70). The three frequency dependent functions l, m, n in (70) must therefore be chosen so that the moments f_2, f_4, g_4 have the time dependence which we have just derived. Self-consistency is only ensured if the functions l, m, n exist. Working with real quantities, the spectral density f is given by

$$f = 2(n + l \cos(2k\eta) - m \sin(2k\eta)) \quad (95)$$

in the radiation cosmos. As one verifies from the definition of g_4 in (49) and of f_0 and f_1 in (48), g_4 can be written

$$g_4 = 4n_4 - f_4. \quad (96)$$

From the definitions of f_2 and f_4 in (48) one obtains together with (92) and (94)

$$\int k^2(l \cos(2k\eta) - m \sin(2k\eta))dk = cb^2\eta^2/2 - n_2, \quad (97)$$

$$\int k^4(l \cos(2k\eta) - m \sin(2k\eta))dk = -cb^2/4. \quad (98)$$

(98) can be obtained from (97) by differentiation with respect to η , so only (97) is needed. All k -integrations considered so far run from $k = 0$ to infinity. We formally extend $l(k)$ and $m(k)$ to negative values by

$$l(k) = l(-k), \quad m(k) = -m(-k) \text{ for } k < 0. \quad (99)$$

This allows us to rewrite (97) as complex Fourier transform

$$\mathcal{F}[\phi(k)|2\eta] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) \exp(2ik\eta) dk = \sqrt{\frac{2}{\pi}} \left(\frac{cb^2\eta^2}{2} - n_2 \right) \quad (100)$$

applied to the complex valued function $\phi(k) = k^2(l(k) + im(k))$. Since $\phi^*(k) = \phi(-k)$, the Fourier transform of $\phi(k)$ is real. To find $\phi(k)$, one has to invoke the Fourier inversion theorem. This requires to consider the right-hand-side of (100) also for negative values of η . Since the right hand side is not absolutely integrable over the whole time axis, $\phi(k)$ must be understood as generalized function (see, e.g., [27], [28], [29] for a confirmation of the subsequent calculations). Extending the function space in this way, the Fourier inversion theorem remains valid. For instance, a polynomial in η gives rise to the Dirac delta function $\delta(k)$ and its derivatives in Fourier space. One obtains

$$k^2 l(k) = -\frac{cb^2}{4} \frac{d^2 \delta}{dk^2} - \frac{n_2}{4\pi} \delta(k) \quad (101)$$

$$k^2 m(k) = 0. \quad (102)$$

We may also extend $n(k)$ to negative values by $n(k) = n(-k)$ for $k < 0$. The spectral components of energy density $\rho(k, \eta)$ and pressure $p(k, \eta)$ are then symmetric functions of k , and the frequency integrated total density can be written $\rho = \frac{1}{2} \int_{-\infty}^{\infty} dk \rho(k, \eta)$, with a similar extension of the integration interval for the pressure. This allows to handle terms involving delta functions applying the usual rules for these functions. Replacing second derivatives of the delta function using the formulae (cf. [27], the prime here denotes the derivative with respect to k)

$$s(k)\delta''(k) = s''(0)\delta(k) - 2s'(0)\delta'(k) + s(0)\delta''(k), \quad (103)$$

one obtains for the spectral decomposition of the energy density and pressure from (52), (53)

$$a^4 G \rho_g(k, \eta) = 2n(k)k^4 - \frac{7}{\eta^2} n(k)k^2 + \frac{7}{4\eta^2} cb^2 \delta''(k) + \left(\frac{14}{\eta^2} n_2 - cb^2 \right) \delta(k), \quad (104)$$

$$3a^4 G p_g(k, \eta) = 2n(k)k^4 - \frac{5}{\eta^2} n(k)k^2 + \frac{5}{4\eta^2} cb^2 \delta''(k) + \left(\frac{10}{\eta^2} n_2 - cb^2 \right) \delta(k). \quad (105)$$

The spectral quantities can be integrated immediately to give the finite total values

$$a^4 G \rho_g = 3a^4 G p_g = 2n_4 - \frac{b^2 c}{2} \quad (106)$$

in agreement with (96), showing the self-consistency of the calculation.

The singularities of the spectral decomposition $\rho_g(k, \eta)$ show that the infrared mode $k = 0$ contributes a *finite* and time-dependent amount $Ga^4 \rho_{ir} = \frac{7}{\eta^2} n_2 - \frac{cb^2}{2}$ to the total energy density. At superhorizon scales, more precisely at scales with $k\eta < \sqrt{3.5}$, the spectral components become negative. Similar conclusions hold for the pressure, which becomes negative for $k\eta < \sqrt{2.5}$. The integrated values of energy density and pressure however stay always positive thanks to the infrared behaviour of their spectral components.

B. $\zeta = 0$: de Sitter scale factor

The vanishing of the second factor in (91) gives the scale factor of the de Sitter universe, $a = \frac{1}{H\eta}$. The density and pressure calculated from (90) are time-independent:

$$\rho_g = -p_g = \frac{3H^2}{8\pi G}. \quad (107)$$

The differential equation for f_2 , (88), has the general solution

$$f_2 = \frac{r_0}{\eta^2} + s_0 \eta^{7/3} - \frac{3}{13\pi H^2} \frac{\ln|\eta|}{\eta^2} \quad (108)$$

with two constants r_0 and s_0 , where r_0 may be gauged to zero. From (86,87) one obtains

$$f_4 = \frac{65}{9} s_0 \eta^{1/3} - \frac{27}{104} \frac{1}{\pi H^2 \eta^4}, \quad (109)$$

$$g_4 = \frac{r_0}{\eta^4} + \frac{61}{9} s_0 \eta^{1/3} - \frac{3}{13} \frac{\ln|\eta|}{\pi H^2 \eta^4} + \frac{33}{104} \frac{1}{\pi H^2 \eta^4}, \quad (110)$$

and the Weyl tensor components are found as

$$\langle C_{0101} \rangle = \langle C_{1212} \rangle = \frac{728}{27} \pi s_0 \eta^{1/3} - \frac{10}{13} \frac{1}{H^2 \eta^4}. \quad (111)$$

We have again to check the compatibility of the spectral density f (defined by (70)) with the time dependence of its moments f_2, f_4, g_4 as given by the last three equations. The integral f_2 corresponding to the definition in (48) is

$$f_2 = 2n_2 + \frac{2}{\eta^2} n_0 + 2 \int_0^\infty k^2 (l \cos(2k\eta) - m \sin(2k\eta)) dk + \frac{2}{\eta^2} \int_0^\infty (-l \cos(2k\eta) + m \sin(2k\eta)) dk - \frac{4}{\eta} \int_0^\infty k (m \cos(2k\eta) + l \sin(2k\eta)) dk. \quad (112)$$

A similar expression holds for f_4 . Straightforward calculation shows that the integrand of $g_4 = \int_0^\infty k^4 \hat{g}(k, \eta) dk$ defined by (49), has the form ($x = k\eta$)

$$\begin{aligned}\hat{g} &= u \cos(2x) + v \sin(2x) + w, \\ u &= \frac{2}{x^4}(-l(x^4 - 3x^2 + 1) + 2mx(x^2 - 1)), \\ v &= \frac{2}{x^4}(m(x^4 - 3x^2 + 1) + 2lx(x^2 - 1)), \\ w &= 2n(1 - 1/x^2 + 1/x^4).\end{aligned}\quad (113)$$

Rewriting the just derived integral expressions for f_2 and g_4 as well as for f_4 as complex Fourier transforms, we obtain:

$$f_2 = 2n_2 + \frac{2}{\eta^2}n_0 + \psi_2 - \frac{1}{\eta^2}\psi_0 + \frac{2i}{\eta}\psi_1, \quad (114)$$

$$f_4 = 2n_4 + \frac{2}{\eta^2}n_2 + \psi_4 - \frac{1}{\eta^2}\psi_2 + \frac{2i}{\eta}\psi_3, \quad (115)$$

$$\begin{aligned}g_4 &= 2n_4 - \frac{2}{\eta^2}n_2 + \frac{2}{\eta^4}n_0 - \psi_4 - \frac{2i}{\eta}\psi_3 \\ &\quad + \frac{3}{\eta^2}\psi_2 + \frac{2i}{\eta^3}\psi_1 - \frac{1}{\eta^4}\psi_0,\end{aligned}\quad (116)$$

where ψ_j is a family of time functions defined by

$$\psi_j = \int_{-\infty}^{\infty} k^j (l + im) \exp(2ik\eta) dk \quad (117)$$

(ψ_2 was already introduced in Section IV). As in the Tolman case we have extended the functions l, m, n to negative values of k by $l(-k) = l(k)$, $m(-k) = -m(k)$ and $n(-k) = n(k)$, thus the functions ψ_j with even (odd) j are real (pure imaginary). The members of a ψ -family are connected by differentiation and integration according to the rule

$$\psi'_j = 2i\psi_{j+1}, \quad (118)$$

which holds also if the ψ_j are generalized functions. The aim is again to obtain the complex spectral density $l(k) + im(k)$ from any of the time functions ψ_j by inverting the corresponding Fourier integral. We start solving (115) for ψ_2 , where f_4 is replaced by the expression (109). Using the rule (118) repeatedly, one obtains the differential equation

$$\begin{aligned}8n_2 + 8n_4\eta^2 + \frac{27}{26\pi H^2\eta^2} - \frac{260}{9}s_0\eta^{7/3} = \\ \eta^2\psi_2'' - 4\eta\psi_2' + 4\psi_2 =\end{aligned}\quad (119)$$

for ψ_2 . Its solution is given by (note that adding a solution of the homogeneous equation would give the wrong time dependence)

$$\psi_2 = 2n_2 - 4n_4\eta^2 + 13s_0\eta^{7/3} + \frac{3}{52\pi H^2}\eta^{-2} \quad (120)$$

If one member of a ψ -family is known, other can be found by differentiation and integration. Straightforward calculation shows, that the ψ -functions derived from (120) satisfy also (114) and (116), if f_2 and g_4 on the left-hand-sides are substituted from (108) and (110). We now apply the Fourier inversion theorem to (117) with $j = 2$ and ψ_2 taken from (120). This requires a continuation of the (real) function ψ_2 into the region $\eta < 0$. Here only the term proportional to s_0 requires more consideration, but this term contributes nothing to ρ_g and p_g , we can therefore put the integration constant s_0 equal to zero. One then obtains

$$k^2 l(k) = n_4 \frac{d^2 \delta(k)}{dk^2} + 2n_2 \delta(k) - \frac{3|k|}{26\pi H^2}, \quad (121)$$

$$k^2 m(k) = 0. \quad (122)$$

Again infrared modes enter the spectral density $l(k)$, but contrary to the Tolman case one is free to specify the spectral function $n(k)$. If $n(k)$ is chosen as zero, the delta function terms in f and in the spectral decomposition of ρ and p vanish, but f has still singular terms seen in an expansion around $k = 0$:

$$f = \frac{3}{13\pi H^2} \left(\frac{1}{k} + \frac{1}{\eta^2 k^3} \right) - \frac{2\eta^4}{39\pi H^2} k^3 + o(k^5) \quad (123)$$

(we missed in section IVC the de Sitter case, since we had excluded infrared singularities). In spite of this singularity, density and pressure of the gravitational waves integrate to the finite constant values (107). Additional terms from a spectral function $n(k) \neq 0$ add nothing to the total values ρ_g and p_g , thus the singular infrared ($k = 0$) component in ρ_g is cancelled by the integrated contribution of n -modes with $k \neq 0$, the same holds for the pressure.

It is easy to extend the calculation to account for a cosmological constant by adding matter terms $\rho_m = \Lambda/(8\pi G)$ and $p_m = -\rho_m$ to the equations (12,13). Repeating the calculation at the beginning of this section, one obtains (for $\zeta = 0$) the same product (91) as in the absence of a Λ constant. The vanishing of the first factor gives a Tolman-de Sitter model, where gravitational radiation has an equation of state (EOS) of the form $p_g = \rho_g/3$ and an energy density decaying as a^{-4} . The scale factor $a(\eta)$ can be expressed in terms of an elliptical integral.

More interesting is the model corresponding to the vanishing second factor, since here the de Sitter scale factor follows. No new calculation is needed: In equation (107) we have to substitute for ρ_g and p_g the total values $\rho_g + \rho_\Lambda$ and $p_g + p_\Lambda$. In the calculations following this equation we must only replace the de Sitter Hubble constant H in all equations by $\tilde{H} = H/(1 - 64\pi^2 G^2 \Lambda/(3H^2))^{1/2}$. The de Sitter expansion is generated by two independent "matter" sources, by a genuine Λ constant as well as by a suitable spectrum of gravitational waves. We may have

an arbitrary (but time-independent) mixture of both ingredients. For sufficiently small Λ gravitons dominate. If Λ reaches the threshold $\Lambda^* = H^2/(64\pi^2 G^2)$, ρ_g becomes zero and turns to negative values for still larger Λ , to ensure the same total energy density for a different composition.

We have not discussed in this article the origin of the primordial wave spectrum, but it should be noted that the expressions for energy density and pressure introduced here may be of interest for concrete models. In the case of a quantum origin due to vacuum fluctuations in a de Sitter cosmos [30], the produced gravitons can be described by a two-point correlation function, which for Bunch-Davies vacuum [31] corresponds to a classical spectral density f given by

$$f_{BD} = \frac{\hbar G}{k\pi^2} \left(1 + \frac{1}{\eta^2 k^2}\right). \quad (124)$$

f_{BD} is obtained by comparing the quantum expectation values of bilinear terms in the metric (as given, e.g., in [32]) with the stochastic averages discussed here. The spectral density f_{BD} has the same infrared singularity as the expression (123) derived previously. Since the coincidence of both spectra holds only approximately for small k , the back reaction of the Bunch-Davies gravitons on the scale factor will change the scale factor[‡]. Again, in spite of the singularity of the spectral density f_{BD} , the integrated values of energy density and pressure require no infrared cut-off. If one introduces an ultraviolet cut-off at the frequency k_1 with $x_1 = k_1\eta$, one obtains for energy density and pressure of the Bunch-Davies gravitons

$$\rho_g = \frac{\hbar H^4}{4\pi^2} x_1^2 (x_1^2 - 7), \quad p_g = \frac{\hbar H^4}{12\pi^2} x_1^2 (x_1^2 + 1). \quad (125)$$

Considering limiting cases, for high frequencies $x_1 \gg 1$ follows the expected EOS $p = \rho/3$, for low frequencies $x_1 \ll 1$ one has $p = -\rho/3$ (together with $\rho < 0$). The expressions (125) for ρ_g and p_g apply only for a de Sitter scale factor. As a result of the back reaction, the local values of ρ_g and p_g change with the background geometry (thus they are, in a sense, not local): The lower the frequency, the stronger is the dependence of the effective EOS on the background gravitational field.

C. $\zeta = 1$: Full back reaction models

Unfortunately, the approach described in the previous subsections does not work in the real case $\zeta = 1$. Even

[‡]Similar conclusions follow from the important work by Tsamis and Woodard, who have discussed *quantum* gravity back reaction on an inflationary expansion rate in numerous papers [8].

the first step, finding analytic solutions of (88),(89) for a, f_2 , has been so far unsuccessful. The back reaction function can be written $b = \zeta(\frac{a''}{a} + \frac{a'^2}{a^2})$ for a dominant gravitational wave background, thus the differential equation for the wave amplitudes h takes for $\zeta = 1$ the unusual form

$$h'' + h(k^2 + \frac{a''}{a} + 2\frac{a'^2}{a^2}) = 0. \quad (126)$$

This equation suggests that graviton creation persists in a wave dominated universe, provided the scale factor is different from $a \sim \eta^{1/3}$. Note that the latter scale factor is inconsistent with the system (88),(89). For a treatment of the general case one has to resort to numerical calculations, which will be discussed elsewhere.

VI. FINAL REMARKS

We have seen that the stochastic back reaction equations treated here form, on the one hand, an apparently self-consistent system of equations with interesting solutions. On the other hand, it is not clear, how far the solutions deviate from true solutions of Einstein's field equations, either for some range of parameters or in some regions of space-time. We shortly discuss what could be done to clarify and to improve the situation.

One has to realize that *ensemble averages* are considered, which always differ from actual realizations of a random process. This is inherent to the method and cannot be changed, but one is able to say something more about statistical deviations from true solutions, if a Gaussian or some other process is assumed.

The main shortcoming of the approach is the use of a second-order approximation to general relativity, but improvements are possible. In principle, Monin and Yaglom's statistical treatment of nonlinear field theories works for arbitrary nonlinearities, if they are present in polynomial form. Only the technical complexity grows in a full treatment, since many higher-order correlation functions must be taken into account. Writing the Einstein field equations as

$$(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\rho\sigma}g_{\rho\sigma})g^3 = \kappa T_{\mu\nu}g^3, \quad (127)$$

where g is the determinant of the metric tensor, we see that the left-hand sides consist of huge polynomials of maximal degree 12 in the (exclusively) covariant components of the metric tensor and its first and second derivatives. Taking the ensemble average of these expressions leads to correlation functions up to sixth order. If the random process $g_{\mu\nu}$ is Gaussian, the higher order correlation functions can be reduced to the second-order functions studied in this article. This would allow us to turn the back reaction equations into -in some sense - exact

relations. Present discussions on a primordial stochastic gravitational wave background usually assume a *quantum* origin of gravitons, which are born out of zero-point vacuum fluctuations. The classical correlations discussed here are expected to be related to quantum mechanical expectation values, thus it seems natural to assume Gaussianity.

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